



ELSEVIER

Available online at www.sciencedirect.com

Linear Algebra and its Applications 421 (2007) 85–96

**LINEAR ALGEBRA
AND ITS
APPLICATIONS**

www.elsevier.com/locate/laa

Isospectral flows that preserve matrix structure

G.M.L. Gladwell^{a,*}, Oscar Rojo^{b,1}

^a *Department of Civil Engineering, University of Waterloo, Ontario, Canada N2L 3G1*

^b *Department of Mathematics, Universidad Católica del Norte, Antofagasta, Chile*

Received 21 March 2005; accepted 30 March 2006

Available online 6 June 2006

Submitted by S. Kirkland

Abstract

The matrix $A = (a_{ij}) \in S_n$ is said to lie on a strict undirected graph \mathcal{G} if $a_{ij} = 0$ ($i \neq j$) whenever (i, j) is not in $\mathcal{E}(\mathcal{G})$. If S is skew-symmetric, the isospectral flow $\dot{A}(t) = [A, S]$ maintains the spectrum of A . We consider isospectral flows that maintain a matrix $A(t)$ on a given graph \mathcal{G} . We review known results for a graph \mathcal{G} that is a (generalised) path, and construct isospectral flows for a (generalised) ring, and a star, and show how a flow may be constructed for a general graph. The analysis may be applied to the isospectral problem for a lumped-mass finite element model of an undamped vibrating system. In that context, it is important that the flow maintain other properties such as irreducibility or positivity, and we discuss whether they are maintained. © 2006 Elsevier Inc. All rights reserved.

AMS classification: 15A18; 15A24; 15A29; 15A90

Keywords: Isospectral flow; Toda flow; Staircase matrix; Structured matrix

1. Introduction

We begin by placing the problem in a physical context.

A discrete undamped vibrating system has natural frequencies $(\omega_i)_1^n$ that are (the square roots of) the eigenvalues $(\lambda_i)_1^n$ of a matrix equation

$$(K - \lambda M)u = 0. \quad (1)$$

* Corresponding author.

E-mail address: ggladwell@uwaterloo.ca (G.M.L. Gladwell).

URL: <http://civil.uwaterloo.ca/ggladwell> (G.M.L. Gladwell).

¹ Project Fondecyt 1040218, Chile.

Here [8] K , M are the stiffness and mass (inertia) matrices of the system; both are symmetric; K is positive semi-definite, positive definite if the system is anchored; M is positive definite. We thus define a system as a matrix pair (K, M) . *Inverse eigenvalue problems* concern the construction of a system (K, M) , or of all systems, having a specified spectrum $(\lambda_i)_1^n$. *Isospectral problems* concern the construction of a system (K, M) , or of all systems, having the same spectrum as a given system (K_0, M_0) . Implicit in these problems is the condition that the matrices K , M have specified *structures*. The term *structure* refers to the pattern of non-zero entries in the matrix, to the signs of the entries and maybe the minors of the matrix. Isospectral problems are somewhat easier to solve than inverse eigenvalue problems: at least, for the former, one does not have to consider the problem of existence, for there is at least one system, namely (K_0, M_0) , with the specified structure. For inverse problems, there may be no system (K, M) , with a specified structure, that has the given spectrum. For simplicity, we shall assume that the mass of the system is *lumped*, so that the mass matrix M is diagonal, and Eq. (1) may easily be reduced to standard form

$$(A - \lambda I)x = 0 \quad (2)$$

by taking $M = D^2$, $A = D^{-1}KD^{-1}$, $Du = x$.

We begin our study by recalling some concepts from graph theory, and their counterparts in terms of matrix structure.

2. Graphs and matrices

Following Tutte [10] we say that a graph \mathcal{G} is *strict* if it has no multiple joins or loops. It is said to be *undirected* if there is no preferred direction for its edges. Henceforward we use the term *graph* to mean a strict undirected graph. We denote the vertex and edge sets of \mathcal{G} by $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ respectively. A graph \mathcal{G} is said to be *connected* if one can go from any vertex i to any other vertex j along a chain of edges in $\mathcal{E}(\mathcal{G})$. If \mathcal{G} is not connected it may be dissected into a number of connected subgraphs.

We denote the set of square matrices of order n by M_n , and the subset of symmetric matrices by S_n . A matrix $A \in S_n$ is said to be a matrix *on* \mathcal{G} if $a_{ij} = 0$ whenever $i \neq j$ and vertices i , j of \mathcal{G} are not joined by an edge, i.e., (i, j) is not in $\mathcal{E}(\mathcal{G})$. This definition says nothing about the diagonal entries a_{ii} , nor about whether $a_{ij} \neq 0$ when $(i, j) \in \mathcal{E}(\mathcal{G})$. A matrix A is said to be a matrix *strictly on* \mathcal{G} if it is on \mathcal{G} and $a_{ij} \neq 0$ if $(i, j) \in \mathcal{E}(\mathcal{G})$; \mathcal{G} is said to be the *strict graph* of A . A matrix $A \in S_n$ is said to be *reducible* if there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where A_1 and A_2 are square matrices. Otherwise it is called *irreducible*. It is known that A is irreducible iff its strict graph is connected. Note that the operation $A \rightarrow PAP^T$ corresponds to relabelling the vertices of the graph \mathcal{G} underlying A .

If $A \in S_n$, its *spectrum* $\sigma(A)$ is defined to be its set of eigenvalues $(\lambda_i)_1^n$. Two matrices $A, B \in S_n$ are said to be *isospectral* if they have the same spectrum: $\sigma(A) = \sigma(B)$. It is known that A, B are isospectral iff there is an orthogonal matrix Q such that $B = Q^T A Q$.

As stated in the abstract, our concern in this paper is to find families of isospectral symmetric matrices on a given graph \mathcal{G} .

There are three important simple graphs: the *path*, the *circuit* (or *ring*) and the *star*; their accompanying matrices are the tridiagonal or *Jacobi* matrix, the *periodic Jacobi* matrix, and the *bordered* matrix.

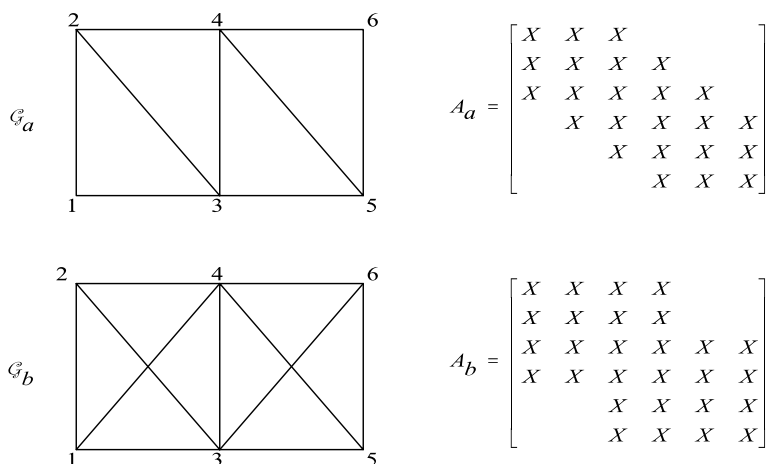


Fig. 1. Simple FEM graphs and their corresponding matrices.

A graph \mathcal{G} is said to be a *clique* if each pair of vertices in \mathcal{G} is joined by an edge; if it has m vertices then it has $m(m-1)/2$ edges, and it is called an m -clique. Cliques appear in the finite element method (FEM); Fig. 1 shows a graph \mathcal{G}_a consisting of four 3-cliques; \mathcal{G}_a may be considered to be a path in which each vertex has been replaced by a 3-clique. The corresponding matrix A_a is an example of a *staircase matrix*; a Jacobi matrix is another example.

The definition of a staircase matrix $A \in S_n$ is as follows. A sequence

$$\rho = \{\rho(1), \rho(2), \dots, \rho(n)\}$$

is a *staircase sequence* if it is non-decreasing and satisfies $\rho(i) \geq i$, $i = 1, 2, \dots, n$. A matrix $A \in S_n$ is said to be a ρ -staircase matrix if, for $j > i$, $a_{i,j} = 0$ whenever $j > \rho(i)$. The matrix A_a is a staircase matrix with $\rho = \{3, 4, 5, 6, 6, 6\}$. We say that

$$\rho' = \{\rho'(1), \rho'(2), \dots, \rho'(n)\}$$

is an *under-staircase sequence* if it is non-decreasing and satisfies $\rho'(i) \leq i$, $i = 1, 2, \dots, n$. If $A \in S_n$ is a staircase matrix then there is an under-staircase sequence ρ' such that, for $j > i$, $a_{i,j} = 0$ whenever $i < \rho'(j)$. For A_a , $\rho' = \{1, 1, 1, 2, 3, 4\}$. Clearly, if $\rho(i) = i$ for some i , $1 \leq i \leq n-1$, then A is reducible. Similarly, A is reducible if $\rho'(i) = i$ for some i , $2 \leq i \leq n$.

\mathcal{G}_b is a graph composed of two 4-cliques; again it corresponds to a staircase matrix. Of course A_a and A_b may be viewed as a pentadiagonal matrix or band matrix with half band width 2, and a 2×2 block tridiagonal matrix, respectively.

Cliques assembled in a ring, as in \mathcal{G}_c , shown in Fig. 2 will correspond to a *periodic staircase matrix*.

Such a matrix contains a central staircase, and staircase-like arrays in its upper right and lower left corners. For such a matrix $A \in S_n$ there is now a second staircase sequence $\gamma = \{\gamma(1), \gamma(2), \dots, \gamma(n)\}$ such that $\gamma(i) > \rho(i)$, and $a_{i,j} = 0$ if $\rho(i) < j < \gamma(i)$. There is also a complementary non-decreasing sequence γ' with $\gamma'(i) < \rho'(i)$, and $a_{i,j} = 0$ if $\gamma'(j) < i < \rho'(j)$. There is a further condition on the sequences; for a periodic staircase matrix, it is as if the matrix were inscribed on a cylinder, so that when the cylinder is cut, the matrix structures in the corners appear adjacent to the opposite sides, as depicted in Fig. 3. For the matrix on the cut cylinder to be a staircase, we must have $\gamma'(n) \leq \rho(1)$, $\gamma(1) \geq \rho'(n)$.

In this case, the four sequences are

$$\begin{aligned} \rho &= \{3, 4, 5, 6, 6, 6\}, & \gamma &= \{5, 6, 7, 7, 7, 7\}, \\ \rho' &= \{1, 1, 1, 2, 3, 5\}, & \gamma' &= \{0, 0, 0, 0, 1, 2\}. \end{aligned}$$

The graphs $\mathcal{G}_a, \mathcal{G}_b, \mathcal{G}_c$ correspond to cliques arranged along a path or a ring; in a FEM analysis of a two-dimensional structure such as a membrane, as in \mathcal{G}_d , shown in Fig. 4, the matrix is block tridiagonal because each layer of the structure is linked only to its immediately neighbouring layers. In each layer, there is a path of cliques, so that each block is tridiagonal or bidiagonal. In a FEM analysis of acoustic vibrations in 3-D, the underlying matrix will be block tridiagonal; each block would itself be tridiagonal; and each block again being tridiagonal or maybe bidiagonal.

We now consider the problem of starting with a matrix A on a certain graph \mathcal{G} and changing it so that it retains the same spectrum, but remains on \mathcal{G} .

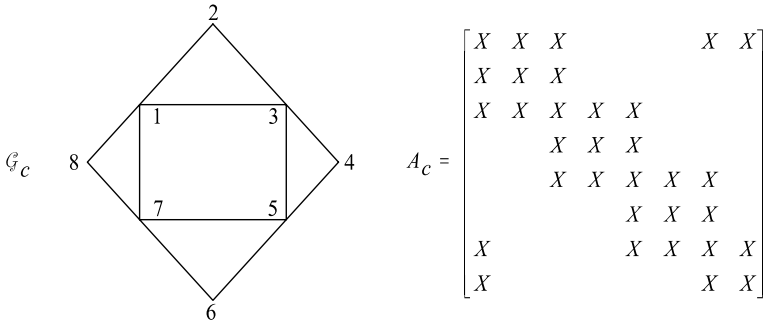


Fig. 2. A ring of cliques and the corresponding periodic staircase matrix.

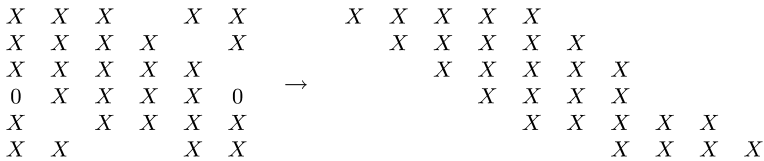


Fig. 3. A periodic staircase matrix viewed as a staircase inscribed on a cylinder.

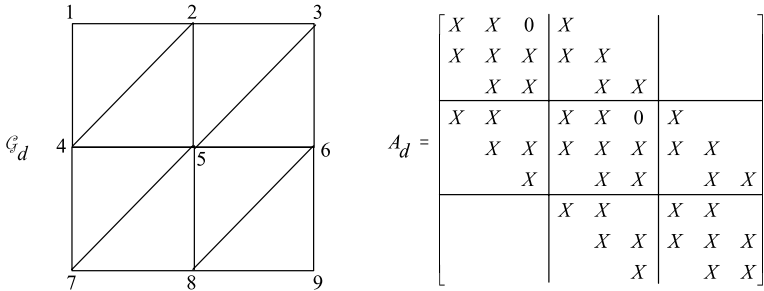


Fig. 4. A FEM graph for a 2-D structure; its corresponding matrix is block tridiagonal.

3. Some simple isospectral flows

If a matrix $A \in S_n$ depends on a parameter t and varies in such a way that its spectrum $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ remains constant, then

$$A = Q \wedge Q^T,$$

where $Q = Q(t)$ is orthogonal and $\wedge = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. Thus, if $\dot{\cdot} = \frac{d}{dt}$, then

$$\dot{A} = Q \wedge \dot{Q}^T + \dot{Q} \wedge Q^T = (Q \wedge Q^T) Q \dot{Q}^T + \dot{Q} Q^T (Q \wedge Q^T).$$

Since Q is orthogonal, $Q Q^T = I$, so that

$$Q \dot{Q}^T + \dot{Q} Q^T = 0.$$

Thus $Q \dot{Q}^T = S$, $\dot{Q} Q^T = -S$ and

$$\dot{A} = AS - SA = [A, S]. \quad (3)$$

The matrix S is skew-symmetric because

$$S^T = (Q \dot{Q}^T)^T = \dot{Q} Q^T = -S.$$

The flow (3) is called *isospectral* flow. Most of the applications of isospectral flow relate to what is generally called *Toda* flow, in which

$$S = A^+ - A^{+T},$$

where A^+ denotes the upper triangle of A . (We always interpret the upper (lower) triangle as the strict upper (lower) triangle.) We note that this S may be rewritten by using the entry-wise Hadamard product:

$$S = A \cdot T, \quad (4)$$

where

$$t_{i,j} = \begin{cases} +1 & \text{if } i < j, \\ 0 & \text{if } i = j, \\ -1 & \text{if } i > j. \end{cases} \quad (5)$$

In general, we suppose S depends on A : $S = S(A)$.

Toda flow is intimately related to QR decomposition and reversal, i.e., to the sequence

$$A_{k-1} = Q_k R_k, \quad A_k = R_k Q_k, \quad k = 1, 2, \dots$$

For historical accounts of this analysis, see Chu [4], Watkins [11], or Chu and Norris [5]. Arbentz and Golub [1] proved two important results for the sequence

$$A_{k-1} - \sigma_k I = Q_k R_k, \quad A_k = \sigma_k I + R_k Q_k, \quad k = 1, 2, \dots$$

1. If $A = A_0 \in S_n$ is reducible through the permutation matrix P , then all QR iterates are reducible by means of the same P .
2. If $A \in S_n$ is irreducible, then the zero pattern of A is preserved by the QR algorithm iff A is a staircase matrix.

The first result has an immediate analogue for Toda flow: if $A(0)$ is reducible through the permutation P , then $A(t)$ is reducible by means of the same P :

$$PAP^T = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \text{implies} \quad PSP^T = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$$

(but note that S_i is not $A_i^+ - A_i^{+T}$) and Eq. (3) splits into

$$P\dot{A}P^T = \begin{bmatrix} PAP^T, PSP^T \end{bmatrix} = \begin{bmatrix} [A_1, S_1] & 0 \\ 0 & [A_2, S_2] \end{bmatrix}.$$

This means that without loss of generality we may confine our attention to irreducible $A \in S_n$, i.e., to connected graphs \mathcal{G} .

In the early papers it was shown that Toda flow maintains tridiagonal (i.e., Jacobi) form; later it was shown that it maintains staircase form. In a recent paper Gladwell [7] showed that Toda flow maintains positivity properties, including total positivity, TP; non-singular total non-negativity, NTN; oscillatory, O.

In order to introduce the analysis, we recall some results from graph theory. Given a graph \mathcal{G} , we may form its *complement* $\overline{\mathcal{G}}$; this is the graph on the same vertex set $\mathcal{V}(\mathcal{G})$, but is such that two vertices of $\overline{\mathcal{G}}$ are adjacent iff they are not adjacent in \mathcal{G} . If $A \in S_n$ and \mathcal{G} is a graph with $\mathcal{V}(\mathcal{G}) = \{1, 2, \dots, n\}$, then we may partition $A = A_1 + A_2$ where A_1 is on \mathcal{G} , A_2 is on $\overline{\mathcal{G}}$, and the diagonal entries of A are placed in A_1 . We may now write Eq. (3) as

$$\dot{A}_1 + \dot{A}_2 = [A_1 + A_2, S] = [A_1, S] + [A_2, S]. \quad (6)$$

Now choose S so that $[A_1, S]$ is on \mathcal{G} , i.e.,

$$[A_1, S]_2 = 0. \quad (7)$$

If $A(0)$ is on \mathcal{G} , so that $A_2(0) = 0$, then Eq. (6) shows that

$$\dot{A}_2 = [A_2, S]_2, \quad A_2(0) = 0,$$

which has the unique solution $A_2(t) = 0$: $A(t)$ is on \mathcal{G} , i.e., $A = A_1$, and

$$\dot{A}_1 = [A_1, S]_1. \quad (8)$$

If $S = A \cdot T$ for *any* (skew-symmetric) T , then

$$S = (A_1 + A_2) \cdot T = A_1 \cdot T + A_2 \cdot T = S_1 + S_2$$

and we may replace (7) by the simpler

$$[A_1, S_1]_2 = 0 \quad (9)$$

for then Eq. (6) show that

$$\begin{aligned} \dot{A}_2 &= [A_1, S_2]_2 + [A_2, S]_2, \quad A_2(0) = 0 \\ &= [A_1, A_2 \cdot T]_2 + [A_2, S]_2, \end{aligned}$$

so that again the unique solution with $A_2(0) = 0$ is $A_2(t) = 0$.

To illustrate this, consider Toda flow. Suppose ρ is a staircase sequence, with ρ' being the corresponding understaircase sequence. Now $S = A \cdot T$ where T is given in (5). Let $[A_1, S_1]_2 = B$; if (i, j) is in the upper triangle $i < j$ then, since S_1 has the same staircase structure as A_1 , we have

$$b_{ij} = \sum_{k=\rho'(j)}^{\rho(i)} (a_{i,k}s_{k,j} - s_{i,k}a_{k,j}).$$

If $\rho'(j) > \rho(i)$, the sum is empty. Otherwise, $i < \rho'(j)$ and $j > \rho(i)$, so that $j > \rho(i) \geq k$ and $i < \rho'(j) \leq k$. This means that (k, j) and (i, k) lie in the upper triangle; thus $s_{k,j} = a_{k,j}$ and $s_{i,k} = a_{i,k}$, and each term in the sum is zero; A maintains staircase form.

Henceforth, we rename Toda flow, *Toda-S* flow (S for staircase) to distinguish it from variants.

Now suppose that A is a periodic staircase matrix, and consider the flow *Toda-P* (P for periodic) given by (3) and (4), where T is skew-symmetric and, in the upper triangle, $t_{i,j} = +1$ on the staircase, and $t_{i,j} = -1$ on the corner structure. In the upper triangle $[A_1, S_1]_2 = B$ is given by

$$b_{ij} = \sum_{k=1}^{\gamma'(j)} (a_{i,k}s_{k,j} - s_{i,k}a_{k,j}) + \sum_{k=\rho'(j)}^{\rho(i)} (a_{i,k}s_{k,j} - s_{i,k}a_{k,j}) + \sum_{k=\gamma(i)}^n (a_{i,k}s_{k,j} - s_{i,k}a_{k,j}).$$

Again, any of the sums is empty if the lower limit exceeds the upper. In the first sum $k \leq \gamma'(j) \leq j$ means that (k, j) is in the upper corner and $s_{k,j} = -a_{k,j}$; $k \leq \gamma'(j) < i \leq \rho(i)$ means that (i, k) is in the lower part of the staircase, and $s_{i,k} = -a_{i,k}$; each term in the sum is zero. The second sum is zero as before. In the third sum $k \geq \gamma(i) > j \geq \rho'(j)$ means that (k, j) is in the lower part of the staircase, and $s_{k,j} = -a_{k,j}$; while $k \geq \gamma(i) \geq i$ means that (i, k) is in the upper corner, and $s_{i,k} = -a_{i,k}$; each term in the sum is zero. Thus $B = 0$, and A retains periodic staircase form.

We noted that a periodic staircase matrix could be viewed as a staircase matrix inscribed on a cylinder. Since the upper corner of A appears to the left, and so below the diagonal, it is fitting that $s_{ij} = -a_{ij}$ there; similarly the lower corner appears to the right, and so above the diagonal, and $s_{ij} = +a_{ij}$ there.

The authors could not find any reference to *Toda-P* in the literature. The flow is important because of its simplicity. In Section 1, we discussed inverse eigenvalue and isospectral problems. The inverse eigenvalue problem for a Jacobi matrix is a classical problem with roots going back to Gantmakher and Krein [6] and to Lanczos [9]; see Boley and Golub [3] and Gladwell [8] for surveys. It has a supremely elegant solution which links the problem to the construction of a sequence of orthogonal polynomials, three-term recurrence relations, moment problems, etc. See Gladwell [8]. By contrast, the inverse eigenvalue problem for a periodic Jacobi matrix has only a complicated, inelegant solution, Boley and Golub [2] and Xu [12]. It is thus noteworthy that there is a simple isospectral flow, *Toda-P*. It is an open question as to whether *Toda-P* has an algebraic counterpart, in the same way as $QR \rightarrow RQ$ is to *Toda-S*.

4. Some more general isospectral flows

In the preceding analysis, we specified S and showed that it preserved certain structures e.g., *Toda-S* for staircase, *Toda-P* for periodic staircase. We shall say that a skew-symmetric S is on \mathcal{G} if the symmetric matrix $S^+ + S^-$ is on \mathcal{G} . In this section, we lift any restrictions on the form of S , asking only that it be skew-symmetric. We pose two problems:

Problem 1. Given a graph \mathcal{G} , find skew-symmetric matrices S , if any, such that the flow (3) maintains the matrix $A(t)$ on \mathcal{G} .

Problem 2. Given a graph \mathcal{G} , find skew-symmetric matrices S on \mathcal{G} , if any, such that the flow (3) maintains the matrix $A(t)$ on \mathcal{G} .

Thus Toda- S and Toda- P solve Problem 2 (and hence Problem 1) for A being respectively a staircase and a periodic staircase.

As we shall see, in general, Problem 2 has no solution, while Problem 1 does have a solution, in principle, for arbitrary \mathcal{G} . Note that whether or not Problems 1 and 2 have solutions is independent of the way in which the vertices of \mathcal{G} are labelled.

As we showed in Section 2, the equation governing S is Eq. (7), which may be written

$$[A_1, S_1]_2 + [A_1, S_2]_2 = 0. \quad (10)$$

Consider this equation in the upper triangle $i < j$; suppose there are p entries in S_1 , q in S_2 , where $p + q = n(n-1)/2 = N$. List the entries in S_1 and S_2 in two vectors $s_1 \in M_{p,1}$ and $s_2 \in M_{q,1}$. The q equations in (10) have the form

$$M_1 s_1 + M_2 s_2 = 0, \quad (11)$$

where $M_1 \in M_{q,p}$, $M_2 \in M_{q,q}$.

First, we make some general statements. If $\text{rank}(M_1) < p$, then Eq. (11) has a solution with $s_2 = 0$; there is a solution to Problem 2. This is the case for the staircase and the periodic staircase. For these cases, we identified particular solutions, Toda- S and Toda- P ; below, we investigate *how many* solutions there are to Problem 2, for a band matrix and a periodic band matrix.

If $M = (M_1, M_2)$, and $\text{rank}(M) = r$, then Eq. (11) has $N - r$ linearly independent solutions, and in general, these will provide solutions to Problem 1. If $\text{rank} M_1 = p$, then Problem 2 has no solution. To obtain a solution of Problem 1, we note that there are two cases: if $\text{rank}(M_2) < q$ then there is the solution $s_1 = 0, s_2 : S$ lies on $\overline{\mathcal{G}}$. This situation occurs in some singular cases such as case a) for the bordered matrix appearing below. Otherwise, $\text{rank}(M_2) = q$, M_2 is invertible. We list the entries of A_1 in the upper triangle in a vector $a_1 \in M_{p,1}$, and there is the solution

$$s_1 = a_1, \quad s_2 = -M_2^{-1} M_1 a_1. \quad (12)$$

We call this the Toda- C (C for complement) solution. We conclude that Problem 1 always has a solution. We explore these solutions for some particular cases.

Suppose A_1 is a *band matrix* of half width b , i.e., there are b non-zero diagonals on either side of the principal diagonal. We pose Problem 2, i.e., we consider Eq. (9), and ask how many solutions there are. Now

$$p = \sum_{m=1}^b (n-m) = \frac{b}{2}(2n-b-1),$$

and $q = N - p$. However, some of Eq. (9) will be trivially satisfied because there is no overlap between row (A_1) and column (S_1), or row (S_1) and column (A_1). The only equations that will not be satisfied trivially are those pertaining to the *next* b diagonals. There are thus

$$k = \sum_{m=b+1}^{2b} (n-m) = \frac{b}{2}(2n-3b-1)$$

non-zero rows of M_1 , provided that $n > (3b+1)/2$, and it can readily be verified that these rows are linearly independent; thus $\text{rank}(M_1) = k$, and

$$p - k = \frac{b}{2}(2n-b-1) - \frac{b}{2}(2n-3b-1) = b^2.$$

In particular, therefore, for a Jacobi matrix ($b = 1$), there is only one solution of $M_1 s_1 = 0$; it must be $s_1 = a_1$, Toda- S .

Generally, for $b > 1$, there are b^2 solutions to Problem 2. Thus for $n = 4$, $b = 2$ there is just one equation

$$a_{1,2}s_{2,4} + a_{1,3}s_{3,4} - s_{1,2}a_{2,4} - s_{1,3}a_{3,4} = 0$$

linking the five unknowns $s_{1,2}$, $s_{1,3}$, $s_{2,4}$, $s_{3,4}$; there are four solutions, the first of which is Toda- S

$$\begin{bmatrix} 0 & a_{1,2} & a_{1,3} & 0 \\ & 0 & a_{2,3} & a_{2,4} \\ \text{skew} & & 0 & a_{3,4} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 & a_{1,3} & a_{1,2} & 0 \\ & 0 & a_{2,3} & a_{3,4} \\ \text{skew} & & 0 & a_{2,4} \\ & & & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -a_{3,4} & a_{2,4} & 0 \\ & 0 & a_{2,3} & a_{1,3} \\ \text{skew} & & 0 & -a_{1,2} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 \\ \text{skew} & & 0 & 0 \\ & & & 0 \end{bmatrix}.$$

Now suppose A_1 is a periodic band matrix. Now $p = nb$. Again, the only non-trivial equations in (9) are those pertaining to the *next* b diagonals adjoining the central staircase and the *next* b adjoining the corner staircase. Since these two sets of diagonals may overlap, we must consider two cases:

- $2b + 1 \leq n - 2b$. There is no overlap between the two sets of extra diagonals, and there are nb equations for nb unknowns. We know that there is one solution, Toda- P , so that $\text{rank}(M_1) < nb$. In fact, as may easily be verified $\text{rank}(M_1) = nb - 1$, so that there is just one solution, Toda- P .
- $2b + 1 > n - 2b$. Now there is overlap, and there are only $n(n - 2b - 1)/2$ equations in nb unknowns: there are $n(4b - n + 1)/2$ solutions, one of which is Toda- P . For example, when $n = 4$, $b = 1$, there are two solutions

$$\begin{bmatrix} 0 & a_{1,2} & 0 & -a_{1,4} \\ & 0 & a_{2,3} & 0 \\ \text{skew} & & 0 & a_{3,4} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 & a_{3,4} & 0 & -a_{2,3} \\ & 0 & a_{1,4} & 0 \\ \text{skew} & & 0 & a_{1,2} \\ & & & 0 \end{bmatrix}.$$

When $n = 6$, $b = 2$, there are nine solutions, one of which is Toda- P ; another typical solution is

$$\begin{bmatrix} 0 & a_{4,5} & a_{4,6} & 0 & -a_{2,4} & -a_{3,4} \\ & 0 & a_{5,6} & a_{1,5} & 0 & -a_{3,5} \\ & & 0 & a_{1,6} & a_{2,6} & 0 \\ & & & 0 & a_{1,2} & a_{1,3} \\ \text{skew} & & & & 0 & a_{3,3} \\ & & & & & 0 \end{bmatrix}.$$

Suppose A is a *bordered* matrix

$$A = \begin{bmatrix} a_1 & b_2 & b_3 & \dots & b_n \\ b_2 & a_2 & & & \\ b_3 & & a_3 & & \\ \vdots & & & & \\ b_n & & & & a_n \end{bmatrix}.$$

Now $p = n - 1$ and $q = n(n - 1)/2 - (n - 1) = (n - 1)(n - 2)/2$, and $s_1 = \{s_{1,2}, s_{1,3}, \dots, s_{1,n}\}$, $s_2 = \{s_{2,3}, s_{2,4}, \dots, s_{n-1,n}\}$. Eqs. (7) are

$$b_i s_{1,j} + b_j s_{1,i} + (a_i - a_j) s_{i,j} = 0, \quad i = 2, 3, \dots, n-1; \quad j = i+1, \dots, n.$$

There are some special solutions for $n = 3, 4$; these we will examine later. There is always the Toda-C solution

$$s_{1,i} = b_i, \quad 2b_i b_j + (a_i - a_j) s_{i,j} = 0.$$

In order for $s_{i,j}$ to be properly defined, we need to ensure that $a_i - a_j$ cannot become zero. We should also investigate whether b_i can become zero, for in that case A will become reducible. The differential equations governing the a_i , b_i are obtained from Eq. (8). Assuming for a moment that all the $a_i - a_j$ are non-zero, we have

$$\dot{a}_1 = -2 \sum_{i=2}^n b_i^2, \quad \dot{a}_j = 2b_j^2, \quad j = 2, \dots, n, \quad (13)$$

$$\dot{b}_j = b_j \left(a_1 - a_j + 2 \sum_{k=2}^n ' \frac{b_k^2}{a_j - a_k} \right), \quad j = 2, \dots, n, \quad (14)$$

where $'$ denotes $k \neq j$.

Consider the product

$$P = \prod_{j=2}^n b_j \prod_{\substack{j,k=2 \\ j < k}}^n (a_j - a_k)$$

and use the product rule in the form:

$$\text{if } f = \prod f_i \quad \text{then } \frac{\dot{f}}{f} = \sum \frac{\dot{f}_i}{f_i}.$$

We have

$$\frac{\dot{P}}{P} = \sum_{j=2}^n \frac{\dot{b}_j}{b_j} + \sum_{\substack{j,k=2 \\ j < k}}^n \frac{\dot{a}_j - \dot{a}_k}{a_j - a_k}.$$

Eqs. (13) and (14) give

$$\frac{\dot{P}}{P} = \sum_{j=2}^n \left(a_1 - a_j + \sum_{k=2}^n ' \frac{b_k^2}{a_j - a_k} \right) + \sum_{\substack{j,k=2 \\ j < k}}^n \frac{2b_j^2 - 2b_k^2}{a_j - a_k} = \sum_{j=2}^n (a_1 - a_j) = g(t). \quad (15)$$

To examine $g(t)$, we note that

$$g(t) = na_1 - \text{tr}(A) = na_1 - \sum_{i=1}^n \lambda_i,$$

and since Eq. (13) shows that $\dot{a}_1 \leq 0$, $g(t) \leq na_1(0) - \sum_{i=1}^n \lambda_i$. On the other hand, if μ is the smallest eigenvalue of A , then $A + |\mu|I$ is positive semi-definite and $a_1 \geq -|\mu|$ and $g(t) \geq -n|\mu| - \sum_{i=1}^n \lambda_i$; $g(t)$ is bounded, and Eq. (15) has the solution

$$P(t) = C \exp \left(\int_0^t g(\tau) d\tau \right),$$

which shows that $P(t)$ maintains the sign it had when $t = 0$. In particular, therefore, if all the $(a_j(0))_2^n$ are distinct, and all the $(b_j(0))_2^n$ are non-zero, then $(a_j(t))_2^n$ will be distinct and the $(b_j(t))_2^n$ will be non-zero.

We now examine the particular solutions, assuming that $(b_j(0))_2^n$ are all non-zero.

(a) $n = 3$, then $p = 2$, $q = 1$ and there is just one equation

$$b_2 s_{1,3} + b_3 s_{1,2} + (a_2 - a_3) s_{2,3} = 0$$

so that $M_1 = [b_3 b_2]$, $M_2 = [a_2 - a_3]$. Since M_1 has rank 1, there is a solution to Problem 2, namely

$$s_{1,2} = b_2, \quad s_{1,3} = -b_3, \quad s_{2,3} = 0,$$

in which

$$\left. \begin{aligned} \dot{a}_1 &= -2b_2^2 + 2b_3^2, & \dot{a}_2 &= 2b_2^2, & \dot{a}_3 &= -2b_3^2 \\ \dot{b}_2 &= (a_1 - a_2)b_2, & \dot{b}_3 &= (a_3 - a_1)b_3; \end{aligned} \right\}$$

A remains irreducible.

There is also another solution, to Problem 1: if $a_2(0) \neq a_3(0)$ it is the Toda- C solution. If $a_2(0) = a_3(0)$ then it is

$$s_{1,2} = 0 = s_{1,3}, \quad s_{2,3} = 1$$

in which

$$\dot{a}_1 = 0 = \dot{a}_2 = \dot{a}_3, \quad \dot{b}_2 = -b_3, \quad \dot{b}_3 = b_2.$$

This is an example of S being on $\overline{\mathcal{G}}$. Since

$$b_2(t) = C \cos(t - \alpha), \quad b_3(t) = C \sin(t - \alpha)$$

there will be instants at which one or other of $b_2(t)$, $b_3(t)$ will be zero.

(b) $n = 4$, then $p = 3 = q$ and

$$M_1 = \begin{bmatrix} b_3 & b_2 & 0 \\ b_4 & 0 & b_2 \\ 0 & b_4 & b_3 \end{bmatrix}, \quad M_2 = \text{diag}(a_2 - a_3, a_2 - a_4, a_3 - a_4),$$

and $\det M_1 = -2b_2b_3b_4 \neq 0$, so that there is no solution to Problem 2. There are three independent solutions to Problem 1. If $(a_j(0))_2^4$ are all distinct; they may be taken to be

(1)

$$s_{1,i} = b_i, \quad (a_i - a_j)s_{i,j} + 2b_i b_j = 0, \quad i = 2, 3, 4; j > i.$$

This is the Toda- C solution.

(2)

$$\left. \begin{aligned} s_{1,2} &= b_2, & s_{1,3} &= b_3, & s_{1,4} &= -b_4 \\ (a_2 - a_3)s_{2,3} + 2b_2b_3 &= 0, & s_{2,4} &= 0 = s_{3,4}. \end{aligned} \right\}$$

Again, we may show that

$$[b_2b_3(a_2 - a_3)]' = (2a_1 - a_2 - a_3)[b_2b_3(a_2 - a_3)]$$

so that $b_2, b_3, a_2 - a_3$ retains the signs they had when $t = 0$. Similarly $\dot{b}_4 = (a_4 - a_1)b_4$, so that b_4 retains its sign also.

(3)

$$\left. \begin{aligned} s_{1,2} &= b_2, & s_{1,3} &= -b_3, & s_{1,4} &= b_4 \\ (a_2 - a_4)s_{2,4} + 2b_2b_4 &= 0, & s_{2,3} &= 0 = s_{3,4}, \end{aligned} \right\}$$

and we may argue as in (2).

If $a_3(0) = a_4(0) \neq a_2(0)$, then the Toda- C solution may be replaced by

$$\left. \begin{aligned} s_{1,2} &= b_2, & s_{1,3} &= 0 = s_{1,4} \\ (a_2 - a_3)s_{2,3} + b_2b_3 &= 0 = (a_2 - a_4)s_{2,4} + b_2b_4 \end{aligned} \right\}$$

and we may argue as before that $a_3(t) = a_4(t) = a_3(0)$, and $a_2(t) \neq a_4(t)$.

If $a_2(0) = a_3(0) = a_4(0)$, then $s_{1,2} = 0 = s_{1,3} = s_{1,4}$, and $s_{2,3}, s_{2,4}, s_{3,4}$ are arbitrary; S is on \mathcal{G} .

- (c) $n \geq 5$. Now $p < q$ and $\text{rank}(M_1) = p$, so that there is no solution to Problem 2. If $(a_j(0))_2^n$ are distinct, then $\text{rank} M = q$, and there are p linearly independent solutions to Problem 1, one of which is the Toda- C solution. The special cases in which two or more of the $(a_j(0))_2^n$ are equal, may be treated as before.

We conclude that, for this simple case of a bordered matrix, corresponding to \mathcal{G} being a star, the Toda- C solution maintains the fundamental characteristics of irreducibility, and the distinctness of $(a_j(t))_2^n$.

5. Future directions

Very often in FEM analysis, the underlying matrices are required to maintain some permutational symmetry, an important particular case being centrosymmetry. There is a need to construct isospectral flows that maintain centrosymmetry. We have shown that it is possible to find an isospectral flow, Toda- C , that maintains a matrix on an arbitrary graph \mathcal{G} . It is necessary to investigate the qualitative properties of such flows, to investigate whether there is a range of values on t for which the flow is regular.

References

- [1] P. Arbenz, G.H. Golub, Matrix shapes invariant under the symmetric QR algorithm, *Numer. Linear Algebra Appl.* 2 (1995) 87–93.
- [2] D. Boley, G.H. Golub, A modified method for reconstructing periodic Jacobi matrices, *Math. Comp.* 42 (1984) 143–150.
- [3] D. Boley, G.H. Golub, A survey of matrix inverse eigenvalue problems, *Inverse Problems* 3 (1987) 595–622.
- [4] M.T. Chu, The generalized Toda flow, the QR algorithm and the center manifold theory, *SIAM J. Algebr. Discrete Methods* 5 (1984) 187–201.
- [5] M.T. Chu, L.K. Norris, Isospectral flows and abstract matrix factorizations, *SIAM J. Matrix Anal. Appl.* 25 (1988) 1383–1391.
- [6] F.P. Gantmakher, M.G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, AMS Chelsea Publishing, Providence, RI, 2002 (this translation of the 1950 Russian version lists the first author as Gantmacher).
- [7] G.M.L. Gladwell, Total positivity and Toda flow, *Linear Algebra Appl.* 350 (2002) 279–284.
- [8] G.M.L. Gladwell, *Inverse Problems in Vibration*, Kluwer Academic Publishers, Dordrecht, 2004.
- [9] C. Lanczos, An iteration method for the solution of the eigenvalue problem of linear differential and integral operators, *J. Res. Nat. Bur. Standards* 45 (1950) 255–282.
- [10] W.T. Tutte, *Graph Theory*, Addison Wesley, Reading, MA, 1984.
- [11] D.S. Watkins, Isospectral flows, *SIAM Rev.* 26 (1984) 379–391.
- [12] S.F. Xu, *An Introduction to Inverse Algebraic Eigenvalue Problems*, Vieweg, Braunschweig, 1998.